NORMANHURST BOYS HIGH SCHOOL

# MATHEMATICS EXTENSION 2

2022 Year 12 Course Assessment Task 4 (Trial Examination)
Thursday 14 August, 2022

**General instructions**

- Working time – 3 hours.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer grid provided (on page 13)

**SECTION II**

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**NESA STUDENT #:** .......................... **# BOOKLETS USED:** .....

**Class** (please ✔)

◯ 12MXX.1 – Miss Ham

◯ 12MXX.2 – Mr Sekaran

Marker's use only.

| **QUESTION** | 1-10 | 11 | 12 | 13 | 14 | 15 | 16 | % |
|---|---|---|---|---|---|---|---|---|
| **MARKS** | $\overline{10}$ | $\overline{14}$ | $\overline{12}$ | $\overline{15}$ | $\overline{17}$ | $\overline{17}$ | $\overline{15}$ | $\overline{100}$ |

# Section I

**10 marks**
**Attempt Question 1 to 10**
**Allow approximately 15 minutes for this section**

Mark your answers on the answer grid provided (labelled as page 13).

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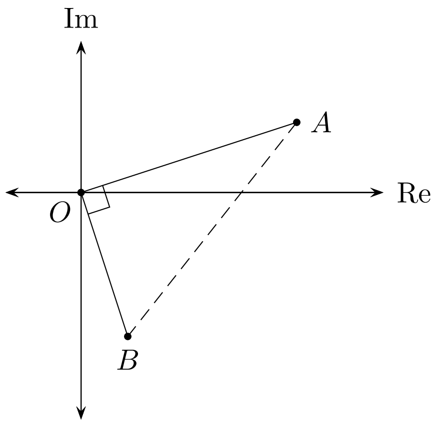
**Questions** **Marks**

**1.** The Argand diagram shows $\triangle OAB$ where $A$ represents the complex number $3e^{i\frac{\pi}{10}}$, $\angle BOA = 90°$, and $\left|\overrightarrow{OB}\right| = \dfrac{2}{3}\left|\overrightarrow{OA}\right|$. **1**




Which of the following represents the complex number $B$?

(A) $2e^{-i\frac{2\pi}{5}}$

(B) $2e^{i\frac{3\pi}{5}}$

(C) $2e^{-i\frac{9\pi}{10}}$

(D) $3e^{i\frac{3\pi}{5}}$

**2.** Let $z = \cos\theta + i\sin\theta$. Which of the following is the expression for $\text{Im}\left(\dfrac{1}{z^5} - \overline{z}^3\right)$? **1**

(A) $\cos 5\theta + \cos 3\theta$

(B) $\cos 5\theta - \cos 3\theta$

(C) $-\sin 5\theta - \sin 3\theta$

(D) $\sin 3\theta - \sin 5\theta$

**3.** Consider the lines $\ell_1 : \underset{\sim}{r_1} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}$ and $\ell_2 : \underset{\sim}{r_2} = \begin{pmatrix} 6 \\ -2 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$. **1**

Which of the following points do $\ell_1$ and $\ell_2$ intersect at?

(A) $(3, -5, 5)$

(B) $(5, 1, -3)$

(C) $(7, 4, -7)$

(D) $\ell_1$ and $\ell_2$ are skew

**4.** A particle moves in a straight line and its motion is described by **1**

$$v^2 = 6(7 - 12x - 2x^2)$$

where $x$ is its displacement from the origin in metres and $v$ is its velocity in $\text{ms}^{-1}$.

Which of the following statements is true?

(A) Frequency is 12 ; Centre of motion is $x = -3$

(B) Frequency is 24 ; Centre of motion is $x = 3$

(C) Frequency is $2\sqrt{3}$ ; Centre of motion is $x = -3$

(D) Frequency is $2\sqrt{6}$ ; Centre of motion is $x = 3$

**5.** The points $P$, $Q$ and $R$ are collinear where $\overrightarrow{OP} = \underset{\sim}{i} - \underset{\sim}{j}$, $\overrightarrow{OQ} = -3\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OR} = 2\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}$ for some constants $m$ and $n$. **1**

Which of the following are possible values for $m$ and $n$?

(A) $m = -1$ and $n = -1$

(B) $m = -1$ and $n = 1$

(C) $m = 1$ and $n = 1$

(D) $m = 1$ and $n = -1$

**6.** Which of the following is the negation of $p \Rightarrow (q \wedge r)$? **1**

(A) $\sim p \Rightarrow (q \wedge r)$

(B) $(\sim q \vee \sim r) \Rightarrow \sim p$

(C) $\sim p \wedge (q \wedge r)$

(D) $p \wedge (\sim q \vee \sim r)$

**7.** Which of the following expressions is equivalent to $\displaystyle\int_0^{\frac{\pi}{6}} e^{\frac{x}{2}} \sin 3x \; dx$? **1**

(A) $\displaystyle -6 - 2\int_0^{\frac{\pi}{6}} e^{\frac{x}{2}} \sin 3x \; dx$

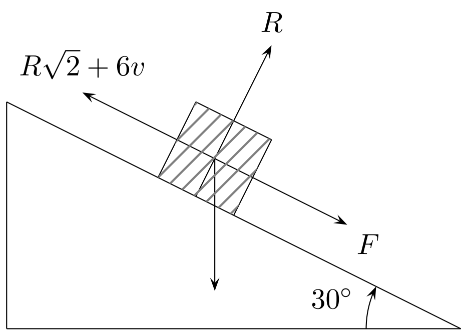
(B) $\displaystyle 2e^{\frac{\pi}{12}} - 6\int_0^{\frac{\pi}{6}} e^{\frac{x}{2}} \cos 3x \; dx$

(C) $\displaystyle 6 + 2\int_0^{\frac{\pi}{6}} e^{\frac{x}{2}} \sin 3x \; dx$

(D) $\displaystyle \frac{1}{2}e^{\frac{\pi}{12}} - \frac{3}{2}\int_0^{\frac{\pi}{6}} e^{\frac{x}{2}} \cos 3x \; dx$

**8.** An object of mass 50 kg is pulled by a constant force of $F$ newtons, down a long rough slope inclined at $30^\circ$ to the horizontal. **1**

The object is met with a total resistive force of $R\sqrt{2} + 6v$, where $R$ is the normal reaction force exerted by the slope on the object and $v$ is the velocity of the object in $\text{ms}^{-1}$. The acceleration of the object along the slope is $a$ $\text{ms}^{-2}$ and the acceleration due to gravity is $g$ $\text{ms}^{-2}$.




Which of the following is an expression for $F$?

(A) $F = 50a - 25g + R\sqrt{2} + 6v$

(B) $F = 50a - 25\sqrt{3}g + R\sqrt{2} + 6v$

(C) $F = -R\sqrt{2} - 6v$

(D) $F = 25g + R\sqrt{2} + 6v$

**9.** Which of the following statements is FALSE? **1**

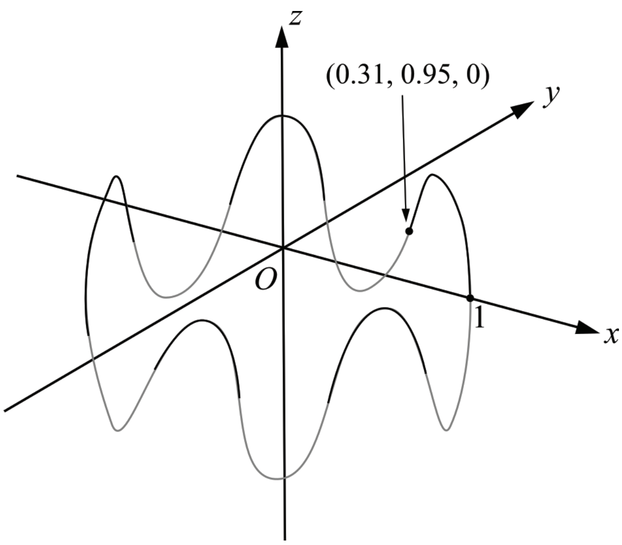
(A) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}$ such that $\sqrt{x} < e^y$

(B) $\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R}$ such that $|y| < x$

(C) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $\cos x = \cos(-y)$

(D) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}^+$ such that $x^4 = 2y^3$

**10.** A curve in three dimensions is represented by vector $\underset{\sim}{r}(t)$ as shown in the diagram below **1**




Which of the following is a possible vector for $\underset{\sim}{r}(t)$?

(A) $\underset{\sim}{r}(t) = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + \sin(5t)\,\underset{\sim}{k}$

(B) $\underset{\sim}{r}(t) = \sin(5t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} + \sin(t)\,\underset{\sim}{k}$

(C) $\underset{\sim}{r}(t) = \cos(5t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j} + \cos(t)\,\underset{\sim}{k}$

(D) $\underset{\sim}{r}(t) = \sin(t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} + \cos(5t)\,\underset{\sim}{k}$

**Examination continues overleaf. . .**

# Section II

**90 marks**
**Attempt Questions 11 to 16**
**Allow approximately 2 hours and 45 minutes for this section.**

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (14 Marks) Commence a NEW booklet. **Marks**

(a) i. Find the Cartesian equation defined by **2**

$$\Big|z - 2(1+i)\Big| = \Big|z + 6(1+i)\Big|$$

ii. Hence sketch the region on the Argand diagram defined by **1**

$$\Big|z - 2(1+i)\Big| \leq \Big|z + 6(1+i)\Big|$$

(b) Solve for $w$ given that $w^3 = 32\sqrt{3} - 32i$. **3**

(c) Find $\alpha$ and $\beta$ given that $z^3 + 9z + 6\sqrt{3}i = (z-\alpha)(z-\beta)^2$. **3**

(d) Consider the complex number $z = \tan\alpha + i$ where $0 < \alpha < \frac{\pi}{2}$.

i. Show that $z = \sec\alpha \times e^{\left(\frac{\pi}{2}-\alpha\right)i}$. **2**

ii. Hence find all values of $\alpha$ for which $z^{-2}$ is purely imaginary. **3**

**Examination continues overleaf...**

**Question 12** (12 Marks) Commence a NEW booklet. **Marks**

(a) By expressing as a sum of partial fractions, find **3**

$$\int \frac{7x^2 - 3}{(x^2 + 6x - 3)(2x - 1)}\, dx$$

(b) Use an appropriate substitution to find **4**

$$\int \frac{3}{2 - \sin 2x}\, dx$$

(c) i. Show that $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$. **3**

ii. Hence, or otherwise, evaluate **2**

$$\int_0^{\frac{1}{2}} x \cos^{-1} x\, dx$$

**Examination continues overleaf...**

**Question 13** (15 Marks) Commence a NEW booklet. **Marks**

(a) i. If $I_n = \int_1^e (1-\ln x)^n \, dx$ for $n \in \mathbb{Z}^+$, show for $n \geq 1$ that **2**

$$I_n = -1 + nI_{n-1}$$

ii. Use the principle of mathematical induction to prove that **4**

$$I_n = n!e - 1 - \sum_{r=1}^{n} \frac{n!}{(n-r)!}$$

for all positive integers $n \geq 1$.

(b) A particle on a spring is moving horizontally in simple harmonic motion and $x$ cm is its horizontal displacement from the origin. The particle is initially at its centre of motion and is moving to the right. Its acceleration is given by

$$\ddot{x} = -\frac{x}{16} - \frac{1}{4}$$

The particle has an amplitude of 8 cm and frequency $n$.

i. State the maximum displacement of the particle. **1**

ii. The particle has velocity $v$ cms$^{-1}$. By integral methods, show that **3**

$$v^2 = \frac{1}{16}\left(64 - (x+4)^2\right)$$

iii. Find the maximum speed of the particle. **1**

The displacement of the particle is given by $x = 8\sin(nt) + k$, where $t$ is the time in seconds after it begins moving right from its centre of motion.

iv. State the value of $n$ and $k$. **1**

v. Find the times when the magnitude of the particle's acceleration is the greatest for $0 \leq t \leq 8\pi$. **2**

vi. Hence find when the particle has travelled a total distance of 40 cm. **1**

**Question 14** (17 Marks) Commence a NEW booklet. **Marks**

(a) Prove that if $a,\ b \in \mathbb{R}$, **2**

$$\frac{5}{8}a^4 + \frac{2}{5}b^2 \geq a^2 b$$

(b) Prove that there is no complex number which satisfies the equation **2**

$$|3z| - 3z = -\frac{1}{2}i$$

(c) i. By considering the expansion of $(x+y)^2$, or otherwise, prove the triangle inequality $|x+y| \leq |x| + |y|$ for $x, y \in \mathbb{R}$. **2**

ii. Hence, or otherwise prove that $\Big||x| - |y|\Big| \leq |x-y|$. **2**

(d) Suppose that $a, b, c \in \mathbb{Z}$. Consider the following proposition

*If all $a$, $b$ and $c$ are odd integers, then the equation $ax^2 + bx + c = 0$ has no integer solutions.*

i. State the contrapositive to the proposition. **1**

ii. Hence prove the proposition by proving the contrapositive. **3**

(e) Given that $p,\ q \in \mathbb{Z}$, prove the proposition **5**

*$p^3 + q^3$ is odd iff either only $p$ is odd or only $q$ is odd*

**Examination continues overleaf**...

**Question 15** (17 Marks) Commence a NEW booklet. **Marks**

(a) Consider the line $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ and the point $F(1,3,5)$.

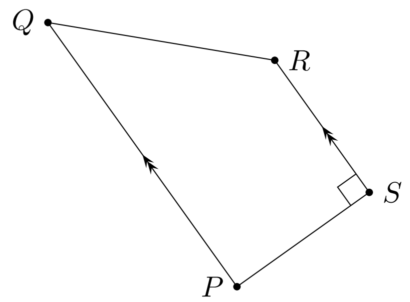
i. $L(-2,3,4)$ is a point on $\underset{\sim}{r}$. Show that the projection of $\overrightarrow{LF}$ along $\underset{\sim}{r}$ is **2**

$$\text{proj}_{\underset{\sim}{r}}\overrightarrow{LF} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

ii. Hence find the coordinates of $E(x,y,z)$ such that $\left|\overrightarrow{FE}\right|$ is the shortest distance from $F$ to the line $\underset{\sim}{r}$. **2**

(b) Consider the points $A(1,4,2m)$ and $B(0,2,-2)$. Find $m$ given that $\angle AOB = \dfrac{\pi}{3}$. **3**

(c) The points $P(a,b,c)$, $Q(15,6,12)$, $R(3,5,10)$ and $S(-2,4,3)$ form a trapezium.




It is given that $SP \perp SR$, and $PQ \parallel SR$.

i. Show that $5a + b + 7c = 15$. **2**

ii. Hence find the coordinates of $P$, using vector methods. **3**

(d) Consider the sphere $S$, centred at point $C(5,-1,2)$ with radius $\sqrt{17}$. Line $\ell_1$ intersects the sphere $S$ at points $G$ and $H$, and has parametric equations

$$\begin{cases} x = 1 - 2\mu \\ y = 2 + \mu \\ z = -3 - 2\mu \end{cases} \qquad \text{where } \mu \in \mathbb{R}$$

i. Show that $G(3,1,-1)$ and $H(\frac{25}{3}. -\frac{5}{3}, \frac{13}{3})$ are the points of intersection. **3**

The perpendicular from the centre of a circle to a chord bisects the chord.
(Do NOT prove this)

Line $\ell_2$ is parallel to line $\ell_1$ and touches the sphere $S$ at a single point $D$.
It is given that $CD \perp \ell_2$ and that $CD$ intersects chord $GH$.

ii. Using vector methods, find one possible set of coordinates for $D$. Write the coordinates as exact values. **2**

**Examination continues overleaf. . .**

**Question 16** (15 Marks) Commence a NEW booklet. **Marks**

(a) Let $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

i. Show that $a + \dfrac{1}{a} \geq 2$. **1**

It is given that

$$a^{n+1} + \frac{1}{a^{n+1}} + a^{n-1} + \frac{1}{a^{n-1}} = \left(a + \frac{1}{a}\right)\left(a^n + \frac{1}{a^n}\right) \qquad \text{(Do NOT prove this)}$$

ii. Use the principle of mathematical induction to prove that **3**

$$a^n + \frac{1}{a^n} \geq a^{n-1} + \frac{1}{a^{n-1}}$$

for all positive integers $n \geq 1$.

(b) An object of mass 5 kg moves along a horizontal surface subject to a resistance force of magnitude $\frac{1}{6}\sqrt{49 + 2v}$ newtons, where $v$ is the speed of the object. Initially the object has speed 16 ms$^{-1}$.

i. Let $t = T$ be when the object comes to rest. Find the value of $T$. **2**

ii. Hence find the total distance travelled by the object. **3**

(c) A ball of mass $m$ kg is projected vertically upwards from the ground.

While in the air, the ball experiences a force due to air resistance of $\frac{3}{250}mgv^2$ newtons, where $g$ is the acceleration due to gravity, and $v$ is the velocity of the object in metres per second.

i. Show that the acceleration of the ball measured in the upwards direction from its point of projection is given by **1**

$$\ddot{x} = -\frac{1}{250}g\left(250 + 3v^2\right)$$

ii. The ball is projected with an initial velocity $u$ ms$^{-1}$. Show that it reaches a maximum displacement of **3**

$$\frac{125}{3g}\ln\left(\frac{250 + 3u^2}{250}\right)$$

After the ball reaches its maximum height, it begins to fall towards its point of projection.

iii. Find an expression for the acceleration of the ball measured in the downwards direction as it falls. **1**

iv. Hence find terminal velocity $V$ of the ball when it falls, giving the answer as an exact value. **1**

**End of paper.**

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# Sample Band E4 Responses

## Section I

**1.** (A) **2.** (D) **3.** (B) **4.** (C) **5.** (C) **6.** (D) **7.** (B) **8.** (A) **9.** (B) **10.** (D)

## Section II

### Question 11 (Ham)

(a) i. (2 marks)

✓ [1] for modulus of both sides

✓ [1] for final answer

Let $z = x + yi$.

$$\begin{aligned} \left|z - 2(1+i)\right| &= \left|z + 6(1+i)\right| \\ \left|(x-2) + (y-2)i\right| &= \left|(x+6) + (y+6)i\right| \\ \sqrt{(x-2)^2 + (y-2)^2} &= \sqrt{(x+6)^2 + (y+6)^2} \\ x^2 - 4x + y^2 - 4y + 8 &= x^2 + 12x + y^2 + 12y + 72 \\ -16x - 16y - 64 &= 0 \end{aligned}$$

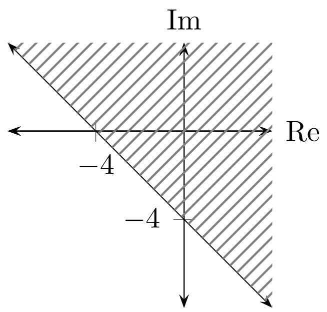
$$\therefore y = -x - 4$$

ii. (1 mark)

✓ [1] for final region

$$\begin{aligned} \left|z - 2(1+i)\right| &\le \left|z + 6(1+i)\right| \\ -16x - 16y - 64 &\le 0 \end{aligned}$$

$$\therefore y \ge -x - 4$$



(b) (3 marks)

✓ [1] for modulus and argument of $w^3$

✓ [1] for one value of $w$

✓ [1] for final solutions

Let $w = re^{i\theta}$.

$$\left|w^3\right| = 64 \quad , \quad \operatorname{Arg}\left(w^3\right) = -\frac{\pi}{6}$$

$$\begin{aligned} w^3 &= 32\sqrt{3} - 32i \\ r^3 e^{3\theta i} &= 64e^{\left(-\frac{\pi}{6}+2\pi k\right)i} \quad , \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

$$r = 4 \quad , \quad \theta = \frac{\pi(-1+12k)}{18}$$

Since $-\pi \le \theta \le \pi$

$$\begin{aligned} -\pi \le \frac{\pi(-1+12k)}{18} &\le \pi \\ -\frac{17}{12} \le k &\le \frac{19}{12} \\ k &= -1, 0, 1 \end{aligned}$$

$$\therefore w = 4e^{-\frac{13\pi}{18}i} \ , \ 4e^{-\frac{\pi}{18}i} \ , \ 4e^{\frac{11\pi}{18}i}$$

(c) (3 marks)

✓ [1] for finding possible values of $\beta$

✓ [1] for final value of $\beta$

✓ [1] for final value of $\alpha$

Let $P(z) = z^3 + 9z + 6\sqrt{3}i$.

$P(\beta) = P'(\beta) = 0$ since $\beta$ is a double root.

$$\begin{aligned} P'(z) &= 3z^2 + 9 \\ P'(\beta) &= 0 \\ 3\beta^2 + 9 &= 0 \\ \beta &= \pm\sqrt{3}i \end{aligned}$$

$$\begin{aligned} P(i\sqrt{3}) &= -3\sqrt{3}i + 9\sqrt{3}i + 6\sqrt{3}i \neq 0 \\ P(-i\sqrt{3}) &= 3\sqrt{3}i - 9\sqrt{3}i + 6\sqrt{3}i = 0 \\ \beta &= -i\sqrt{3} \end{aligned}$$

By taking the sum of roots

$$\alpha + (-i\sqrt{3}) + (-i\sqrt{3}) = 0$$

$$\therefore \alpha = 2\sqrt{3}i \ , \ \beta = -\sqrt{3}i$$

(d) i. (2 marks)

✓ [1] for finding modulus or argument of $z$

✓ [1] for showing final result

$$\begin{aligned} z &= \frac{1}{\cos\alpha}(\sin\alpha + i\cos\alpha) \\ &= \sec\alpha\left(\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)\right) \end{aligned}$$

But $\sec\alpha > 0$ for $0 < \alpha < \frac{\pi}{2}$.

$$\therefore z = \sec\alpha \times e^{\left(\frac{\pi}{2}-\alpha\right)i}$$

ii. (3 marks)

✓ [1] for finding real component

✓ [1] for finding one value

✓ [1] for final values

$$\begin{aligned} z^{-2} &= \cos^2\alpha \cdot e^{(2\alpha-\pi)i} \\ &= \cos^2\alpha \cdot \Big(\cos(2\alpha-\pi) + i\sin(2\alpha-\pi)\Big) \end{aligned}$$

If $z^{-2}$ is purely imaginary, then

$$\begin{aligned} \mathrm{Re}(z^{-2}) &= 0 \\ \cos^2\alpha \cdot \cos(2\alpha-\pi) &= 0 \\ \cos(2\alpha-\pi) &= 0 \\ \cos(\pi-2\alpha) &= 0 \\ -\cos 2\alpha &= 0 \\ 2\alpha &= \frac{\pi}{2} \qquad \text{for } 0 < 2\alpha < \pi \end{aligned}$$

But $\cos^2\alpha \neq 0$
otherwise $z^{-2} = 0$
is purely real

$$\therefore \alpha = \frac{\pi}{4} \text{ only}$$

**Question 12** (Ham)

(a) (3 marks)

✓ [1] for forming simultaneous equations

✓ [1] for finding partial fractions

✓ [1] for final answer

$$\begin{aligned}\frac{7x^2-3}{(x^2+6x-3)(2x-1)} &= \frac{Ax+B}{x^2+6x-3}+\frac{C}{2x-1}\\ 7x^2-3 &= (Ax+B)(2x-1)+C(x^2+6x-3)\end{aligned}$$

By equating coefficients

$$\begin{aligned}7 &= 2A+C\\ 0 &= -A+2B+6C\\ -3 &= -B-3C\end{aligned} \tag{12.1}$$

$$\begin{aligned}A &= \frac{7}{2}-\frac{1}{2}C\\ B &= 3-3C\\ 0 &= -\frac{7}{2}+\frac{1}{2}C+6-6C+6C\\ C &= -5\end{aligned}$$

$$\therefore A=6\ ,\quad B=18\ ,\quad C=-5$$

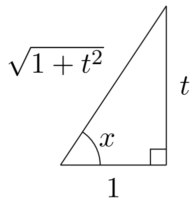
$$\begin{aligned}\int\frac{7x^2-3}{(x^2+6x-3)(2x-1)}\,dx &= \int\frac{6x+18}{x^2+6x-3}+\frac{-5}{2x-1}\,dx\\ &= 3\ln\left|x^2+6x-3\right|-\frac{5}{2}\ln|2x-1|+c\end{aligned}$$

(b) (4 marks)

- ✓ [1] for using $t = \tan x$ finding $dx$ in terms of $t$
- ✓ [1] for substituting $t$-formula and simplifying
- ✓ [1] for completing the square
- ✓ [1] for final answer

Let $t = \tan x$.

$$\begin{aligned}\frac{dt}{dx} &= \sec^2 x \\ \cos^2 x\, dt &= dx \\ \frac{1}{1+t^2}\, dt &= dx\end{aligned}$$




Substituting via $t$-formula

$$\begin{aligned}\int \frac{3}{2-\sin 2x}\, dx &= \int \frac{3}{\left(2 - \frac{2t}{1+t^2}\right)} \times \frac{1}{1+t^2}\, dt \\ &= \frac{3}{2}\int \frac{1}{1+t^2-t}\, dt \\ &= \frac{3}{2}\int \frac{1}{\frac{3}{4} + \left(t-\frac{1}{2}\right)^2}\, dt \\ &= \frac{3}{2}\int \frac{4}{3+(2t-1)^2}\, dt \\ &= \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c \\ &= \sqrt{3}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c \\ &= \sqrt{3}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + c\end{aligned}$$

(c) i. (3 marks)

✓ [1] for an appropriate substitution

✓ [1] for finding the integral

✓ [1] for final result

Let $x = \sin\theta$.

$\theta = \sin^{-1} x$

$$\frac{dx}{d\theta} = \cos\theta \qquad\qquad x = \frac{1}{2} \ \rightarrow\ \theta = \frac{\pi}{6}$$
$$dx = \cos\theta\, d\theta \qquad\qquad x = 0 \ \rightarrow\ \theta = 0$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx &= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 - \cos 2\theta\, d\theta \\
&= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} \\
&= \frac{1}{2}\left(\left(\frac{\pi}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2}\right) - 0\right)
\end{aligned}$$
$$\therefore \int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

ii. (2 marks)

✓ [1] for integrating by parts

✓ [1] for final answer

$$u = \cos^{-1} x \qquad\qquad v' = x$$
$$u' = -\frac{1}{\sqrt{1-x^2}} \qquad\qquad v = \frac{1}{2}x^2$$

$$\begin{aligned}
\int_0^{\frac{1}{2}} x\cos^{-1} x\, dx &= \left[\frac{1}{2}x^2 \cos^{-1} x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -\frac{1}{2} \cdot \frac{x^2}{\sqrt{1-x^2}}\, dx \\
&= \frac{1}{2}\left(\frac{1}{4} \times \frac{\pi}{3} - 0\right) + \frac{1}{2}\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) \\
&= \frac{\pi}{12} - \frac{\sqrt{3}}{16}
\end{aligned}$$

## Question 13 (Ham)

(a) i. (2 marks)

✓ [1] for integrating by parts

✓ [1] for final result

$$\text{Let}\quad u = (1-\ln x)^n \quad , \quad v' = 1$$
$$u' = -\frac{n}{x}(1-\ln x)^{n-1} \quad , \quad v = x$$

$$\begin{aligned} I_n &= \int_1^e (1-\ln x)^n\,dx \\ &= \Big[x(1-\ln x)^n\Big]_1^e - \int_1^e -n(1-\ln x)^{n-1}\,dx \\ &= 0 - 1 + n\int_1^e (1-\ln x)^{n-1}\,dx \\ \therefore I_n &= -1 + nI_{n-1} \end{aligned}$$

ii. (4 marks)

✓ [1] for base case proof

✓ [1] for applying inductive step

✓ [1] for adjusting sum

✓ [1] for final result

Let the proposition be $P(n)$.

**Base case:** Prove $P(1)$ is true.

$$\begin{aligned} \text{LHS} &= I_1 \\ &= -1 + I_0 \qquad \text{from (i)} \\ &= -1 + \int_1^e (1-\ln x)^0\,dx \\ &= -1 + \Big[x\Big]_1^e \\ &= e - 2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= e - 1 - \frac{1!}{0!} \\ &= e - 2 \end{aligned}$$

$\therefore$ LHS = RHS, and $P(1)$ is true.

**Inductive step:** Assume $P(k)$ is true.

$$I_k = k!e - 1 - \sum_{r=1}^{k} \frac{k!}{(k-r)!}$$

**Prove:** Examine $P(k+1)$.

$$\text{RTP:} \quad I_{k+1} = (k+1)!e - 1 - \sum_{r=1}^{k+1} \frac{(k+1)!}{(k+1-r)!}$$

$$\begin{aligned}
\text{LHS} &= -1 + (k+1)I_k \qquad \text{from (i)} \\
&= -1 + (k+1)\left(k!e - 1 - \sum_{r=1}^{k} \frac{k!}{(k-r)!}\right) \qquad \text{from inductive step} \\
&= (k+1)!e - 1 - (k+1) - \sum_{r=1}^{k} \frac{(k+1)!}{(k-r)!} \\
&= (k+1)!e - 1 - (k+1) - \sum_{r=2}^{k+1} \frac{(k+1)!}{\big(k-(r-1)\big)!} \\
&= (k+1)!e - 1 - (k+1) - \left(\sum_{r=1}^{k+1} \frac{(k+1)!}{(k+1-r)!} - \frac{(k+1)!}{(k+1-1)!}\right) \\
&= (k+1)!e - 1 - (k+1) - \sum_{r=1}^{k+1} \frac{(k+1)!}{(k+1-r)!} + (k+1) \\
&= (k+1)!e - 1 - \sum_{r=1}^{k+1} \frac{(k+1)!}{(k+1-r)!}
\end{aligned}$$

$\therefore$ LHS = RHS, and $P(k+1)$ is true if $P(k)$ is true.

Hence by mathematical induction, $P(n)$ is true for all positive integers $n \geq 1$.

(b) i. (1 mark)

✓ [1] for final answer

$$\ddot{x} = -\frac{1}{16}(x+4)$$

Centre of motion: $x_0 = -4$ , Amplitude: $a = 8$

$$\therefore x_{max} = 4$$

ii. (3 marks)

✓ [1] for integrating for $\frac{1}{2}v^2$

✓ [1] for using appropriate $x$ and $v$ values

✓ [1] for final result

$$\begin{aligned} \frac{d\left(\frac{1}{2}v^2\right)}{dx} &= -\frac{1}{16}(x+4) \\ \frac{1}{2}v^2 &= -\frac{1}{16}\int x + 4\, dx \\ -8v^2 &= \frac{1}{2}x^2 + 4x + c \end{aligned}$$

The particle is at rest at its maximum displacement.

$$v = 0 \text{ when } x = 4: \qquad c = -24$$

$$\begin{aligned} -8v^2 &= \frac{1}{2}x^2 + 4x - 24 \\ -16v^2 &= x^2 + 8x - 48 \\ -16v^2 &= (x+4)^2 - 64 \\ \therefore v^2 &= \frac{1}{16}\Big(64 - (x+4)^2\Big) \end{aligned}$$

iii. (1 mark)

✓ [1] for final answer

The particle reaches its maximum speed at the centre of motion.

$$\text{when } x = -4: \qquad \begin{aligned} v^2 &= 4 \\ |v| &= 2 \end{aligned}$$

Hence the maximum speed is 2 $\text{cms}^{-1}$.

iv. (1 mark)

✓ [1] for final answer

Amplitude: $a = 8$

Frequency: $n = \dfrac{1}{4}$

Centre of motion: $x_0 = -4 \quad \rightarrow \quad k = -4$

$$\therefore x = 8\sin\left(\frac{t}{4}\right) - 4$$

v. (2 marks)

✓ [1] for one value of $t$ or setting up correct equations

✓ [1] for final answers

Magnitude of acceleration greatest at its extremeties $x = -12$ and $x = 4$.

$$\begin{aligned} \text{if } x = -12: \qquad -12 &= 8\sin\left(\frac{t}{4}\right) - 4 && \text{for } 0 \le t \le 8\pi \\ \sin\left(\frac{t}{4}\right) &= -1 && \text{for } 0 \le \frac{t}{4} \le 2\pi \\ \frac{t}{4} &= \frac{3\pi}{2} \\ t &= 6\pi \end{aligned}$$

$$\begin{aligned} \text{if } x = 4: \qquad 4 &= 8\sin\left(\frac{t}{4}\right) - 4 && \text{for } 0 \le t \le 8\pi \\ \sin\left(\frac{t}{4}\right) &= 1 && \text{for } 0 \le \frac{t}{4} \le 2\pi \\ \frac{t}{4} &= \frac{\pi}{2} \\ t &= 2\pi \end{aligned}$$

$\therefore |\ddot{x}|$ is a maximum at $t = 2\pi$ and $t = 6\pi$.

vi. (1 mark)

✓ [1] for final answer

For $0 \le t \le 2\pi$, the particle travels 8 cm (to its right extremity).

For $2\pi \le t \le 6\pi$, the particle travels 16 cm (from its right to left extremity).

$\therefore$ By the symmetry of SHM, the particle travels 8 cm per $2\pi$ seconds.

Hence the particle has travelled 40 cm at $t = 10\pi$.

**Question 14** (Ho)

(a) (2 marks)

✓ [1] for appropriate perfect square

✓ [1] for final answer

$$\begin{aligned}(5a^2-4b)^2 &\geq 0\\ 25a^4-40a^2b+16b^2 &\geq 0\\ 25a^4+16b^2 &\geq 40a^2b\\ \therefore \frac{5}{8}a^4+\frac{2}{5}b^2 &\geq a^2b\end{aligned}$$

(b) (2 marks)

✓ [1] for equating real and/or imaginary components

✓ [1] for final answer

Let $z=x+yi$ be the complex solution to the equation, where $x,y\in\mathbb{R}$.

$$\begin{aligned}3\left|x+yi\right|-3x-3yi &= -\frac{1}{2}i\\ 3\sqrt{x^2+y^2}-3x-3yi &= -\frac{1}{2}i\end{aligned}$$

Equating the real components

$$\begin{aligned}3\sqrt{x^2+y^2}-3x &= 0\\ \sqrt{x^2+y^2} &= x\\ x^2+y^2 &= x^2\\ y &= 0\end{aligned}$$

$\therefore z=x$, but $z\in\mathbb{C}$

Hence by proof by contradiction, there is no complex number satisfying the equation.

(c) i. (2 marks)

✓ [1] for applying $xy \le |x||y|$

✓ [1] for final result

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2 \\ &\le |x|^2 + 2|x||y| + |y|^2 \qquad \text{since } xy \le |x||y| \\ (x+y)^2 &\le (|x|+|y|)^2 \\ \sqrt{(x+y)^2} &\le \sqrt{(|x|+|y|)^2} \\ |x+y| &\le \Big||x|+|y|\Big|\end{aligned}$$

$$\therefore |x+y| \le |x|+|y| \qquad \text{since } |x|+|y| \ge 0$$

ii. (2 marks)

✓ [1] for proof of + or − case

✓ [1] for final result

$$\begin{aligned}|x| &= |(x-y)+y| \\ |x| &\le |x-y| + |y| \qquad \text{from (i)} \\ |x| - |y| &\le |x-y|\end{aligned}$$

$$\begin{aligned}\text{Similarly} \qquad |y| - |x| &\le |y-x| \\ -(|x|-|y|) &\le |x-y| \qquad \text{since } |x-y| = |y-x|\end{aligned}$$

If $a \le b$ and $-a \le b$, then $|a| \le b$.

$$\text{Hence } \Big||x|-|y|\Big| \le |x-y|$$

(d) i. (1 mark)

✓ [1] for final answer

*If the equation $ax^2 + bx + c = 0$ has at least one integer solution, then at least one of $a, b, c$ is an even integer.*

(Since if a number is not odd, it is even.)

ii. (3 marks)

✓ [1] for significant progress

✓ [1] for proving one case

✓ [1] for final result

If $ax^2 + bx + c = 0$ has at least one integer solution, at least one solution is either odd or even.

**Case 1:** Assume the equation has at least one odd integer solution:
Let $x = 2m + 1$, where $m \in \mathbb{Z}$.

$$\begin{aligned} a(2m+1)^2 + b(2m+1) + c &= 0 \\ 4am^2 + 4am + a + 2bm + b + c &= 0 \\ a + b + c &= 2k \qquad \text{where } k = -2am^2 - 2am - 2bm \end{aligned}$$

i.e. the sum of $a, b, c$ is even, since $k \in \mathbb{Z}$.

Assume $a, b, c$ are all odd:
Let $a = 2A + 1$, $b = 2B + 1$, $c = 2C + 1$, where $A, B, C \in \mathbb{Z}$.

$$\begin{aligned} a + b + c &= (2A + 1) + (2B + 1) + (2C + 1) \\ &= 2p + 1 \qquad \text{where } p = A + B + C + 1 \end{aligned}$$

i.e. the sum of $a, b, c$ is odd, since $p \in \mathbb{Z}$.

But this contradicts previous result, and so $a, b, c$ cannot all be odd.

$\therefore$ By contradiction, if the equation has at least one odd integer solution, then at least one of $a, b, c$ is even.

**Case 2:** Assume the equation has at least one even integer solution:
Let $x = 2n$, where $n \in \mathbb{Z}$.

$$\begin{aligned} a(2n)^2 + b(2n) + c &= 0 \\ 4an^2 + 2bn + c &= 0 \\ c &= 2q \qquad \text{where } q = -2an^2 - bn \end{aligned}$$

$c$ is even since $q \in \mathbb{Z}$.

$\therefore$ If the equation has at least one even integer solution, then at least one of $a, b, c$ is even.

Hence by the contrapositive, *if all $a$, $b$, $c$ are odd integers, then the equation $ax^2 + bx + c = 0$ has no integer solutions.*

(e) (5 marks)

✓ [1] for stating the proposition and converse
✓ [1] for proving one case of proposition
✓ [1] for proving proposition
✓ [1] for proving one case of converse
✓ [1] for proving converse

**RTP the proposition:** *If either only $p$ is odd or only $q$ is odd, then $p^3 + q^3$ is odd.*

If a number is not odd, it is even.

**Case 1:** Assume only $p$ is odd.
Let $p = 2a + 1$ and $q = 2b$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p^3 + q^3 &= (2a+1)^3 + (2b)^3 \\ &= 8a^3 + 12a^2 + 6a + 8b^3 + 1 \\ &= 2k + 1 \qquad \text{where } k = 4a^3 + 6a^2 + 3a + 4b^3 \end{aligned}$$

$p^3 + q^3$ is odd since $k \in \mathbb{Z}$.

**Case 2:** Assume only $q$ is odd.
Let $p = 2a$ and $q = 2b + 1$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p^3 + q^3 &= (2a)^3 + (2b+1)^3 \\ &= 2k + 1 \qquad \text{similar to above} \end{aligned}$$

$p^3 + q^3$ is odd since $k \in \mathbb{Z}$.

Similarly proved if only $q$ is odd.

$\therefore$ *If either only $p$ is odd or only $q$ is odd, then $p^3 + q^3$ is odd.*

**RTP the converse:** *If $p^3 + q^3$ is odd, then either only $p$ is odd or only $q$ is odd.*

Prove the contrapositive:
*If neither $p$ nor $q$ is odd or if both $p$ and $q$ is odd, then $p^3 + q^3$ is not odd.*

**Case 1:** Assume neither $p$ nor $q$ is odd.
Let $p = 2a$ and $q = 2b$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p^3 + q^3 &= (2a)^3 + (2b)^3 \\ &= 8a^3 + 8b^3 \\ &= 2k \qquad \text{where } k = 4a^3 + 4b^3 \end{aligned}$$

$p^3 + q^3$ is even since $k \in \mathbb{Z}$.

**Case 2:** Assume both $p$ and $q$ is odd. Let $p = 2a + 1$ and $q = 2b + 1$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p^3 + q^3 &= (2a+1)^3 + (2b+1)^3 \\ &= 8a^3 + 12a^2 + 6a + 8b^3 + 12b^2 + 6b + 2 \\ &= 2k \qquad \text{where } k = 4a^3 + 6a^2 + 3a + 4b^3 + 6b^2 + 3b + 1 \end{aligned}$$

$p^3 + q^3$ is even since $k \in \mathbb{Z}$.

$\therefore$ By the contrapositive, *if $p^3 + q^3$ is odd, then either only $p$ is odd or only $q$ is odd.*

Hence $p^3 + q^3$ *is odd iff either only $p$ is odd or only $q$ is odd.*

**Alternate proofs**

**RTP the converse:** *If $p^3 + q^3$ is odd, then either only $p$ is odd or only $q$ is odd.*

$$p^3 + q^3 = (p+q)(p^2 - pq + q^2)$$

$\therefore (p+q)$ and $(p^2 - pq + q^2)$ are both odd, since $p^3 + q^3$ is odd if it has an even factor.

**Case 1:** Assume both $p$ and $q$ are odd.
Let $p = 2a + 1$ and $q = 2b + 1$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p + q &= (2a+1) + (2b+1) \\ &= 2(a+b+1) \\ &= 2k \qquad \text{where } k = a + b + 1 \end{aligned}$$

$\therefore (p+q)$ is even since $k \in \mathbb{Z}$, contradicting previous finding.

**Case 1:** Assume both $p$ and $q$ are even.
Let $p = 2a$ and $q = 2b$, where $a, b \in \mathbb{Z}$.

$$\begin{aligned} p + q &= 2a + 2b \\ &= 2(a+b) \\ &= 2k \qquad \text{where } k = a + b \end{aligned}$$

$\therefore (p+q)$ is even since $k \in \mathbb{Z}$, contradicting previous finding.

Thus either only $p$ is odd or only $q$ is odd, for $(p+q)$ to be odd.

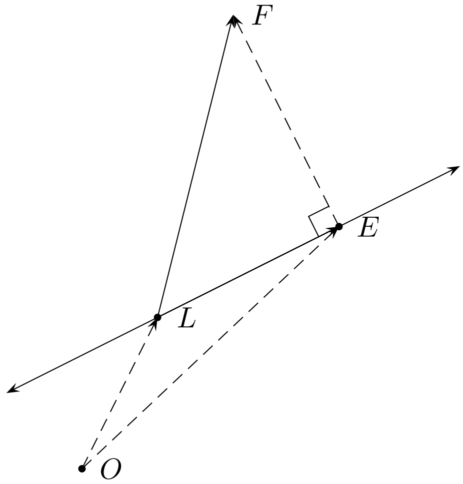
Hence by contradiction, *if $p^3 + q^3$ is odd, then either only $p$ is odd or only $q$ is odd.*

**Question 15** (Sekaran)

(a) i. (2 marks)

✓ [1] for finding $\overrightarrow{LF}$

✓ [1] for final answer




$$\overrightarrow{LF} = \begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}-2\\3\\4\end{pmatrix} = \begin{pmatrix}3\\0\\1\end{pmatrix}$$

Using the direction vector of $\underset{\sim}{r}$:

$$\text{proj}_{\underset{\sim}{r}}\,\overrightarrow{LF} = \frac{\begin{pmatrix}3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}}{(\sqrt{1+4+1})^2} \times \begin{pmatrix}1\\-2\\-1\end{pmatrix}$$

$$\therefore \text{proj}_{\underset{\sim}{r}}\,\overrightarrow{LF} = \frac{1}{3}\begin{pmatrix}1\\-2\\-1\end{pmatrix}$$

ii. (2 marks)

✓ [1] for expression for $\overrightarrow{FE}$

✓ [1] for final answer

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{OL} + \overrightarrow{LE}\\ &= \begin{pmatrix}-2\\3\\4\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\-2\\-1\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}-5\\7\\11\end{pmatrix} \qquad \text{since } \overrightarrow{LE} = \text{proj}_{\underset{\sim}{r}}\,\overrightarrow{LF}\end{aligned}$$

$$\therefore E\left(-\frac{5}{3}, \frac{7}{3}, \frac{11}{3}\right)$$

(b) (3 marks)

- ✓ [1] for correct substitution
- ✓ [1] for forming correct quadratic equation
- ✓ [1] for final answer

$$\begin{aligned}
\overrightarrow{OA}\cdot\overrightarrow{OB} &= \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\frac{\pi}{3}\\
\begin{pmatrix}1\\4\\2m\end{pmatrix}\cdot\begin{pmatrix}0\\2\\-2\end{pmatrix} &= \sqrt{17+4m^2}\times\sqrt{8}\times\frac{1}{2}\\
8-4m &= \sqrt{2(17+4m^2)} \qquad (*)\\
16(4-4m+m^2) &= 2(17+4m^2)\\
4m^2-32m+15 &= 0\\
(2m-1)(2m-15) &= 0
\end{aligned}$$

$$m=\frac{1}{2} \quad \text{or} \quad m=\frac{15}{2}\text{, but } m\le 2 \text{ from } (*)$$

$$\therefore m=\frac{1}{2}\text{ only}$$

(c) i. (2 marks)

- ✓ [1] for finding one vector
- ✓ [1] for final result

$$\overrightarrow{SP}=\begin{pmatrix}a\\b\\c\end{pmatrix}-\begin{pmatrix}-2\\4\\3\end{pmatrix}=\begin{pmatrix}a+2\\b-4\\c-3\end{pmatrix}$$

$$\overrightarrow{SR}=\begin{pmatrix}3\\5\\10\end{pmatrix}-\begin{pmatrix}-2\\4\\3\end{pmatrix}=\begin{pmatrix}5\\1\\7\end{pmatrix}$$

$$\begin{aligned}
\overrightarrow{SP}\cdot\overrightarrow{SR} &= 0 \qquad (\text{since } SP\perp SR)\\
\begin{pmatrix}a+2\\b-4\\c-3\end{pmatrix}\cdot\begin{pmatrix}5\\1\\7\end{pmatrix} &= 0\\
5a+10+b-4+7c-21 &= 0\\
\therefore 5a+b+7c &= 15
\end{aligned}$$

ii. (3 marks)

- ✓ [1] for $\overrightarrow{QP}$ in terms of $\lambda$
- ✓ [1] for finding $\lambda$
- ✓ [1] for final answer

$$\overrightarrow{QP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} 15 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} a-15 \\ b-6 \\ c-12 \end{pmatrix}$$

$$\overrightarrow{RS} = -\begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \qquad \text{from i}$$

$$\begin{aligned} \overrightarrow{QP} &= -\lambda \overrightarrow{RS} \qquad (\text{since } PQ \parallel SR) \\ \begin{pmatrix} a-15 \\ b-6 \\ c-12 \end{pmatrix} &= -\lambda \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \qquad (\text{since } PQ \parallel SR) \end{aligned}$$

Equating the $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$ components

$$\begin{cases} a &= 15 - 5\lambda \\ b &= 6 - \lambda \\ c &= 12 - 7\lambda \end{cases}$$

$$\begin{aligned} 5(15-5\lambda) + (6-\lambda) + 7(12-7\lambda) &= 15 \qquad \text{from i} \\ -75\lambda &= -150 \\ \therefore \lambda &= 2 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} a-15 \\ b-6 \\ c-12 \end{pmatrix} &= -2 \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix} \end{aligned}$$

Hence $P(5, 4, -2)$

(d) i. (3 marks)

✓ [1] for forming $|\underset{\sim}{v} - \underset{\sim}{c}| = \sqrt{17}$

✓ [1] for forming correct quadratic equation

✓ [1] for final result

Any point $P$ on $\ell_1$ has position vector $\underset{\sim}{p}$

$$\underset{\sim}{p} = \begin{pmatrix} 1-2\mu \\ 2+\mu \\ -3-2\mu \end{pmatrix}$$

Any point $V$ on sphere $S$ has position vector $\underset{\sim}{v}$

$$\left| \underset{\sim}{v} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right| = \sqrt{17}$$

$\ell_1$ intersect sphere $S$ when $\underset{\sim}{v} = \underset{\sim}{p}$

$$\begin{aligned} \left| \begin{pmatrix} 1-2\mu \\ 2+\mu \\ -3-2\mu \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right| &= \sqrt{17} \\ \sqrt{(-4-2\mu)^2 + (3+\mu)^2 + (-5-2\mu)^2} &= \sqrt{17} \\ 9\mu^2 + 42\mu + 33 &= 0 \\ 3\mu^2 + 14\mu + 11 &= 0 \\ (3\mu+11)(\mu+1) &= 0 \\ \therefore \mu = -\frac{11}{3} \quad \text{or} \quad \mu &= -1 \end{aligned}$$

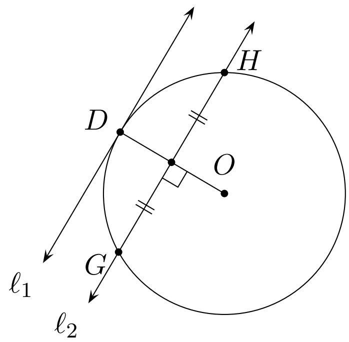
$$\begin{aligned} \text{if } \mu = -\frac{11}{3}: &\qquad \underset{\sim}{p} = \frac{1}{3}\begin{pmatrix} 25 \\ -5 \\ 13 \end{pmatrix} \\ \text{if } \mu = -1: &\qquad \underset{\sim}{p} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \end{aligned}$$

Hence $G(3, 1, -1)$ and $H\left(\frac{25}{3}, -\frac{5}{3}, \frac{13}{3}\right)$.

ii. (2 marks)

✓ [1] for finding unit vector along $vvCD$

✓ [1] for final answer




Let $M$ be the midpoint of $GH$.

$$M = \left(\frac{17}{3}, -\frac{1}{3}, \frac{5}{3}\right)$$

$CM \perp GH$ (given)

$CM \parallel CD$, since $CD \perp \ell_2$ and $\ell_1 \parallel \ell_2$.

$$\overrightarrow{CM} = \frac{1}{3}\begin{pmatrix} 17 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$|\overrightarrow{CM}| = \frac{1}{3}\sqrt{4+4+1} = 1$$

$\overrightarrow{CM}$ is a unit vector along $CD$.

$\overrightarrow{CD}$ is a radius of sphere $S$.

$$\begin{aligned} \overrightarrow{CD} &= \sqrt{17} \times \overrightarrow{CM} \\ \overrightarrow{OD} - \overrightarrow{OC} &= \frac{\sqrt{17}}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \\ \overrightarrow{OD} &= \frac{\sqrt{17}}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

$$\therefore D\left(\frac{2\sqrt{17}}{3} + 5, \frac{2\sqrt{17}}{3} - 1, -\frac{\sqrt{17}}{3} + 2\right)$$

**Question 16** (Sekaran)

(a) i. (1 mark)

✓ [1] for final result

$$\begin{aligned}\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2 &\geq 0\\ a-2+\frac{1}{a} &\geq 0\\ \therefore a+\frac{1}{a} &\geq 2\end{aligned}$$

ii. (3 marks)

✓ [1] for proving base case

✓ [1] for applying given inequality

✓ [1] for final result

Let the proposition be $P(n)$.

**Base case:** Prove $P(1)$ is true.

$$\begin{aligned}\text{LHS} &= a+\frac{1}{a}\\ \text{RHS} &= a^0+\frac{1}{a^0}=2\end{aligned}$$

$$a+\frac{1}{a}\geq 2 \qquad \text{from i}$$

$\therefore$ LHS $\geq$ RHS, and $P(1)$ is true.

**Inductive step:** Assume $P(k)$ is true.

$$a^k+\frac{1}{a^k}\geq a^{k-1}+\frac{1}{a^{k-1}}$$

**Prove:** Examine $P(k+1)$.

$$\text{RTP:} \qquad a^{k+1}+\frac{1}{a^{k+1}}\geq a^k+\frac{1}{a^k}$$

$$a^{k+1}+\frac{1}{a^{k+1}}=\left(a+\frac{1}{a}\right)\left(a^k+\frac{1}{a^k}\right)-\left(a^{k-1}+\frac{1}{a^{k-1}}\right) \qquad \text{from given}$$

$$\begin{aligned}\text{LHS} &= \left(a+\frac{1}{a}\right)\left(a^k+\frac{1}{a^k}\right)-\left(a^{k-1}+\frac{1}{a^{k-1}}\right)\\ &\geq 2\left(a^k+\frac{1}{a^k}\right)-\left(a^{k-1}+\frac{1}{a^{k-1}}\right) \qquad \text{from i}\\ &\geq \left(a^k+\frac{1}{a^k}\right)+\left(a^k+\frac{1}{a^k}\right)-\left(a^{k-1}+\frac{1}{a^{k-1}}\right) \qquad \text{from i}\end{aligned}$$

$$\left(a^k+\frac{1}{a^k}\right)-\left(a^{k-1}+\frac{1}{a^{k-1}}\right)\geq 0 \qquad \text{from inductive step}$$

$$\text{LHS} \geq \left(a^k + \frac{1}{a^k}\right) + 0$$

$\therefore$ LHS $\geq$ RHS, and $P(k+1)$ is true if $P(k)$ is true.

Hence by mathematical induction $P(n)$ is true for all positive integers $n \geq 1$.

(b) i. (2 marks)

✓ [1] for forming correct $m\ddot{x} = -R$ equation

✓ [1] for final answer

$$\begin{aligned}
F &= 5\ddot{x} \\
5\ddot{x} &= -\frac{1}{6}\sqrt{49+2v} \\
\frac{dv}{dt} &= -\frac{1}{30}(49+2v)^{\frac{1}{2}} \\
\int_{16}^{0} (49+2v)^{-\frac{1}{2}}\, dv &= -\frac{1}{30}\int_0^T \, dt \\
\left[\frac{(49+2v)^{\frac{1}{2}}}{\frac{1}{2}\times 2}\right]_{16}^{0} &= -\frac{1}{30}\Big[t\Big]_0^T \\
7 - 9 &= -\frac{1}{30}(T-0) \\
T &= 60
\end{aligned}$$

(i.e. The object comes to rest after 60 seconds.)

ii. (3 marks)

✓ [1] for separating variables

✓ [1] for correctly integrating

✓ [1] for final answer

Let $x = X$ at $t = T$ (i.e. when the object comes to rest).

$$\begin{aligned}
v\frac{dv}{dx} &= -\frac{1}{30}(49+2v)^{\frac{1}{2}} \qquad \text{from i} \\
\int_{16}^{0} \frac{v}{(49+2v)^{\frac{1}{2}}}\, dv &= -\frac{1}{30}\int_0^X \, dx \\
\frac{1}{2}\int_{16}^{0} \frac{49+2v}{(49+2v)^{\frac{1}{2}}} - \frac{49}{(49+2v)^{\frac{1}{2}}}\, dv &= -\frac{1}{30}(X-0) \\
\int_{16}^{0} (49+2v)^{\frac{1}{2}} - 49(49+2v)^{-\frac{1}{2}}\, dv &= -\frac{1}{15}X \\
\left[\frac{(49+2v)^{\frac{3}{2}}}{\frac{3}{2}\times 2} - 49(49+2v)^{\frac{1}{2}}\right]_{16}^{0} &= -\frac{1}{15}X \qquad \text{from i} \\
\left(\frac{7^3}{3} - 329\right) - \left(\frac{9^3}{3} - 441\right) &= -\frac{1}{15}X \\
\therefore X &= 460
\end{aligned}$$

Hence the object travels 460 metres in total.

(c) i. (1 mark)

✓ [1] for final result

$$\begin{aligned} F &= m\ddot{x} \\ m\ddot{x} &= -mg - \frac{3}{250}mgv^2 \\ \therefore \ddot{x} &= -\frac{1}{250}g(250 + 3v^2) \end{aligned}$$

ii. (3 marks)

✓ [1] for correctly integrating

✓ [1] for using $v = 0$

✓ [1] for final result

$$\begin{aligned} v\frac{dv}{dx} &= -\frac{1}{250}g(250 + 3v^2) \qquad \text{from i} \\ \int \frac{v}{250 + 3v^2}\, dv &= -\frac{g}{250}\int dx \\ \frac{1}{6}\ln\left|250 + 3v^2\right| &= -\frac{g}{250}x + c \end{aligned}$$

$x = 0$ and $v = u$ when $t = 0$ :

$$c = \frac{1}{6}\ln(250 + 3u^2)$$

Maximum height at $v = 0$ :

$$\begin{aligned} \frac{1}{6}\ln 250 &= -\frac{g}{250}x + \frac{1}{6}\ln(250 + 3u^2) \\ \frac{g}{250}x &= \frac{1}{6}\ln\left(\frac{250 + 3u^2}{250}\right) \\ \therefore x &= \frac{125}{3g}\ln\left(\frac{250 + 3u^2}{250}\right) \end{aligned}$$

Hence the maximum displacement is $\dfrac{125}{3g}\ln\left(\dfrac{250 + 3u^2}{250}\right)$ metres.

iii. (1 mark)

✓ [1] for final answer

$$\begin{aligned} F &= m\ddot{x} \\ m\ddot{x} &= mg - \frac{3}{250}mgv^2 \\ \therefore \ddot{x} &= \frac{1}{250}g(250 - 3v^2) \end{aligned}$$

iv. (1 mark)

✓ [1] for final answer

$v = V$ when $\ddot{x} = 0$ :

$$\begin{aligned} \frac{1}{250}g(250 - 3V^2) &= 0 \\ 3V^2 &= 250 \\ |V| &= \frac{5\sqrt{10}}{3} \end{aligned}$$

Hence the terminal velocity is $\dfrac{5\sqrt{10}}{3}$ ms$^{-1}$.